Practice problems on Euclidean Geometry and Euclidean Transformations.

**Problem 1.** Let $ABC$ be a right triangle that is oriented clockwise and has angles of $90^\circ$, $30^\circ$, $60^\circ$ at the vertices $A$, $B$, $C$.

(i) Identify $R_{C,120} \circ R_{B,60}$.
(ii) Identify $R_{C,-120} \circ R_{B,-60} \circ R_{A,-180}$.

**Problem 2.** Let $ABCDEF$ be a regular hexagon that is oriented clockwise (so that a rotation from $A$ to $B$ to $C$ to $D$ to $E$ to $F$ is clockwise).

(i) Identify $R_{D,120} \circ R_{A,60}$.
(ii) Identify $R_{F,180} \circ \rho_{ED} \circ R_{D,120}$.

**Problem 3.** Let $ABCDEFGH$ be a regular octagon that is oriented clockwise (so that a rotation from $A$ to $B$ to $C$ to $D$ to $E$ to $F$ to $G$ to $H$ is clockwise). Write $O$ for the center of the octagon.

Identify the following transformations. For each state whether it is a translation, rotation, reflection, or glide reflection. Write down any point, angle or vector needed to identify the transformation. (For points or vectors that do not have names give a diagram to show how they are found.)

(i) $\rho_{GE} \circ \rho_{GA} \circ \rho_{HD}$
(ii) $\rho_{AE} \circ \rho_{BF}$
(iii) $R_{F,135} \circ R_{H,45}$

**Problem 4.** Let $ABC$ be an equilateral triangle that is oriented clockwise (so that a rotation from $A$ to $B$ to $C$ to $A$ is clockwise).

In the following questions, identify any required lines, points, or vectors by giving either a name or a construction using the named points. For example, you might answer that a line is the perpendicular to $BC$ from $A$.

(i) $\rho_{AB} \circ \rho_{BC}$ is a rotation. Identify its center and angle.
(ii) $\rho_{AB} \circ \rho_{BC} \circ \rho_{AB}$ is a reflection. Identify its mirror line.
(iii) $\rho_{AB} \circ \rho_{BC} \circ \rho_{CA}$ is a glide reflection $\gamma_{XY}$. Identify the vector $XY$.

**Problem 5.** Let $ABC$ be a triangle with angles of $30^\circ$, $60^\circ$, and $90^\circ$ at $A$, $B$, and $C$, respectively. Suppose that the triangle is oriented clockwise (so that a rotation from $A$ to $B$ to $C$ to $A$ is clockwise).

In the following questions, identify any required lines, points, or vectors by giving either a name or a construction using the named points. For example, you might answer that a line is the perpendicular to $BC$ from $A$.

(i) $\rho_{AB} \circ \rho_{AC}$ is a rotation. Identify its center and angle.
(ii) $\rho_{AB} \circ \rho_{AC} \circ \rho_{AB}$ is a reflection. Identify its mirror line.
(iii) $\rho_{AB} \circ \rho_{BC} \circ \rho_{CA}$ is a glide reflection $\gamma_{XY}$. Identify the segment $XY$.

**Problem 6.** CHOOSE ONE OPTION ONLY, there is no extra credit for doing both. Either:

State and prove a theorem that describes the (Euclidean) transformations obtained by combining two reflections.

Or:

Let $A = (0,0)$, $B = (0,2)$, $C = (2,0)$. Let $X$ and $Y$ be the midpoints of the $BC$ and $CA$. Identify the following 2 transformations (giving coordinates for any points or vectors that you use).

(i) $R_{A,90} \circ \tau_{BC}$
(ii) $\rho_{AB} \circ \rho_{AC} \circ \rho_{XY}$.

**Answer:**
Problem 7.  **CHOOSE ONE OPTION ONLY**, there is no extra credit for doing both. Either:

State and prove a theorem that shows that (in Euclidean geometry) every isometry can be formed from at most three reflections.

Or:

Let $ABC$ be an equilateral triangle that is oriented clockwise (so that a rotation from $A$ to $B$ to $C$ to $A$ is clockwise). For the following three transformations, identify any required lines, points, or vectors by giving either a name or a construction using the named points.

(i) $\rho_{AB} \circ \rho_{BC}$ is a rotation. Identify its center and angle.
(ii) $\rho_{AB} \circ \rho_{BC} \circ \rho_{AB}$ is a reflection. Identify its mirror line.
(iii) $\rho_{AB} \circ \rho_{BC} \circ \rho_{CA}$ is a glide reflection $\gamma_{XY}$. Identify the vector $XY$.

**Answer:**

Problem 8.  Let $ABCD$ be a square with center $O$ that is oriented counterclockwise (so that a rotation from $A$ to $B$ to $C$ to $D$ is counterclockwise). For the following three transformations, identify any required lines, points, or vectors by giving either a name or a construction using the named points.

(i) $R_A(90^\circ) \circ R_B(90^\circ)$ is a rotation. Identify its center and angle.
(ii) $\rho_{AB} \circ \rho_{AC} \circ \rho_{AB}$ is a reflection. Identify its mirror line.
(iii) $\rho_{BC} \circ \rho_{BD} \circ \rho_{AC}$ is a glide reflection $\gamma_{XY}$. Identify the points $X$ and $Y$.

**Answer:**

Problem 9.  Suppose that $A$, $B$, $C$, and $D$ are the vertices $(0,0)$, $(2,0)$, $(2,2)$, and $(0,2)$ of a square.

(a) The transformation $R_{D,45} \circ R_{A,45}$ is a rotation by $90^\circ$ about a point $P$. Find the coordinates of $P$.

(b) The transformation $\rho_{AC} \circ \gamma_{DA}$ can be decomposed as the combination of reflections across 4 mirror lines. Draw a diagram showing $A$, $B$, $C$, and $D$ and the four mirror lines (marked in order by the numbers 1 to 4).

(c) Two of the mirrors in your answer to (b) can be moved so as to give a set of four mirror lines, three of which are parallel such that the combination of the corresponding reflections has the same result as that of (b). Draw a diagram showing these new mirror lines.

(d) The transformation $\rho_{AC} \circ \gamma_{DA}$ (as in (b) and (c)) simplifies to give a rotation by $90^\circ$ about a center $Q$. Give coordinates for $Q$. 