Problem 1.  
(a) The combination of a clockwise rotation about \((0, 0)\) by 120° followed by a clockwise rotation about \((4, 0)\) by 60° is a rotation. Find the coordinates of its center and its angle of rotation.

Answer:
The two rotations factor as products of reflections, \(\rho_{l_2} \circ \rho_{l_3}\) and \(\rho_{l_4} \circ \rho_{l_5}\), where \(l_1, l_2 = l_3\) and \(l_4\) are the mirror lines shown on the following figure.

The combination is \(\rho_{l_4} \circ \rho_{l_5} \circ \rho_{l_2} \circ \rho_{l_1} = \rho_{l_3} \circ \rho_{l_1}\), which is a rotation centered at \((1, \sqrt{3})\) by an angle of 180°.

(b) Identify the combination formed by first translating by the vector \((2, 0)\) and then rotating by 90° about \((0, 0)\).

Answer:
The translation and rotation factor as products of reflections, \(\rho_{l_2} \circ \rho_{l_1}\) and \(\rho_{l_4} \circ \rho_{l_3}\), where \(l_1, l_2 = l_3\) and \(l_4\) are the mirror lines shown on the following figure.

The combination is \(\rho_{l_4} \circ \rho_{l_3} \circ \rho_{l_2} \circ \rho_{l_1} = \rho_{l_3} \circ \rho_{l_1}\), which is a rotation centered at \((-1, 1)\) by an angle of 180°.

Problem 2.  
Identify the combination formed by first applying the glide reflection \(\gamma_{A,B}\) and then applying \(\gamma_{C,D}\) where \(A = (0, 0), B = (2, 0), C = (1, -2), D = (1, 0)\).

Answer:
The two glide reflections factor as products of reflections, \(\rho_{l_5} \circ \rho_{l_3} \circ \rho_{l_2} \circ \rho_{l_1}\) and \(\rho_{l_6} \circ \rho_{l_5} \circ \rho_{l_4}\), where \(l_1, l_2 = l_6, l_3 = l_4\) and \(l_5\) are the mirror lines shown on the following figure.

The combination is \(\rho_{l_6} \circ \rho_{l_5} \circ \rho_{l_4} \circ \rho_{l_2} \circ \rho_{l_1} = \rho_{l_6} \circ \rho_{l_5} \circ \rho_{l_2} \circ \rho_{l_1} = \rho_{l_5} \circ \rho_{l_6} \circ \rho_{l_2} \circ \rho_{l_1} = \rho_{l_5} \circ \rho_{l_6} \circ \rho_{l_1}\), which is a rotation centered at \((0, 1)\) by an angle of 180°.
Problem 3. A spherical triangle has angles of $120^\circ$, $60^\circ$ and $45^\circ$. Find the cosines of the (arc) lengths of the sides. How many sides have an arc length larger than $90^\circ$?

Answer:
We may apply the second form of the law of cosines:

$$\cos a = \frac{\cos \alpha + \cos \beta \cos \gamma}{\sin \beta \sin \gamma}$$

To find the cosines of the three sides from the three angles.

If the sides opposite the angles $120^\circ$, $60^\circ$ and $45^\circ$ are $x$, $y$ and $z$, respectively, we have:

$$\cos x = \frac{\cos 120 + \cos 60 \cos 45}{\sin 60 \sin 45} = \frac{1 - \sqrt{2}}{\sqrt{3}}$$

$$\cos y = \frac{\cos 60 + \cos 120 \cos 45}{\sin 120 \sin 45} = \frac{\sqrt{2} - 1}{\sqrt{3}}$$

$$\cos z = \frac{\cos 45 + \cos 120 \cos 60}{\sin 120 \sin 60} = \frac{2\sqrt{2} - 1}{3}$$

Observe that only the cosine of $x$ is negative, which means that only $x$ has an obtuse angle measure. This shows that just one side has an arc length that exceeds $90^\circ$. 

(Note that we can interchange the order of $\rho_5$ and $\rho_6$ since their product in either order is a rotation of $180^\circ$ about their intersection point.)
**Problem 4.** Two points on the earth have latitude and longitude coordinates as follows: $A = (30^\circ N, 45^\circ W)$, $B = (60^\circ N, 75^\circ W)$. What direction should a plane fly to follow a great circle route from $A$ to $B$? (Give your answer as the sine of the angle made to the direction of north at $A$.) What is the sine of the (arc) length of the shortest route from $A$ to $B$.

**Answer:**

Let $N$ be the north pole. Then the spherical triangle formed by $A$, $B$, and $N$ is as shown in the following figure.

The angle at $N$ is $75 - 45 = 30^\circ$. The arc lengths of the sides $NA$ and $NB$ are the complements of the latitudes of $A$ and $B$. The length of the arc from $A$ to $B$ is $x$ and the angle of flight (west of north at $A$) is the angle $\alpha$.

We first find $x$ from the law of cosines as applied to the angle at $N$. This gives:

$$
\cos 30 = \frac{\cos x - \cos 30 \cos 60}{\sin 30 \sin 60}
$$

Solving for $\cos x$, we find:

$$
\cos x = \frac{3 + 2\sqrt{3}}{8}
$$

Hence, $\cos^2 x = \frac{21+12\sqrt{3}}{64}$ and $\sin^2 x = \frac{43-12\sqrt{3}}{64}$. This gives

$$
\sin x = \frac{\sqrt{43 - 12\sqrt{3}}}{8}
$$

The law of sines gives

$$
\frac{\sin \alpha}{\sin 30} = \frac{\sin 30}{\sin x}
$$

Hence, $4 \sin \alpha = \frac{8}{\sqrt{43 - 12\sqrt{3}}}$ and the initial angle of flight should be

$$
\sin^{-1} \frac{2}{\sqrt{43 - 12\sqrt{3}}}
$$

west of north.
Problem 5.  (a) Define the polar triangle of a spherical triangle.

Answer:
Each edge of the spherical triangle has two poles. (The poles are the points where the unique diameter of the sphere perpendicular to the plane of the edge meets the sphere). The pole that lies on the same hemisphere as the third vertex of the spherical triangle is the vertex of the polar triangle that corresponds to the edge of the original triangle. The polar triangle is formed by the three vertices picked out in this way by the three edges of the original triangle.

(b) Suppose that a spherical triangle has edges $a$, $b$, $c$ and angles $\alpha$, $\beta$, $\gamma$. If the vectors representing the vertices of the triangle are $A$, $B$, and $C$ give formulas for the vectors that represent the vertices of the polar triangle.

Answer:
We suppose that the original triangle is labeled so that a path from $A$ to $B$ to $C$ and back to $A$ is oriented counterclockwise. If not all formulas given here must be negated.

The vertices of the polar triangle are represented by vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ given by:

$$
\mathbf{a} = (\mathbf{B} \times \mathbf{C})/(\sin a), \quad \mathbf{b} = (\mathbf{C} \times \mathbf{A})/(\sin b), \quad \mathbf{c} = (\mathbf{A} \times \mathbf{B})/(\sin c)
$$

(c) Prove that the polar triangle of the polar triangle is the original triangle.

(The following argument is an alternative to the more geometrical approach used in class.)

We may assume that the original triangle is labeled so that the orientation $ABC$ is counterclockwise. The vertex of the polar triangle of the polar triangle that corresponds to the edge from $B$ to $C$ is the unit vector in the direction of $\mathbf{b} \times \mathbf{c}$. However

$$
\sin b \sin c (\mathbf{b} \times \mathbf{c}) = (\mathbf{C} \times \mathbf{A}) \times (\mathbf{A} \times \mathbf{B}) = ((\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B})\mathbf{A} - ((\mathbf{C} \times \mathbf{A}) \cdot \mathbf{A})\mathbf{B} = ((\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B})\mathbf{A}
$$

this vector has the same direction as the unit vector $\mathbf{A}$. We deduce that $\mathbf{A}$ is a vertex of the polar of the polar triangle and similarly so too are the other two vertices of the original triangle.