Solutions to practice problems on affine geometry.

Problem 1 Find an affine transformation that turns the points $(1,0)$, $(3,5)$, $(2,3)$ into the points $(0,1)$, $(3,2)$, $(4,5)$, respectively. (Give a formula $f(\underline{x}) = M\underline{x} + \underline{b}$ for the transformation.)

Answer:

The transformation that maps $(0,0)$, $(1,0)$, $(0,1)$ to $(1,0)$, $(3,5)$, $(2,3)$ is:

$$P(\underline{x}) = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \underline{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Similarly, the transformation that maps $(0,0)$, $(1,0)$, $(0,1)$ to $(0,1)$, $(3,2)$, $(4,5)$ is:

$$Q(\underline{x}) = \begin{pmatrix} 3 & 4 \\ 1 & 4 \end{pmatrix} \underline{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence, $Q \circ P^{-1}$ maps $(1,0) \mapsto (0,0) \mapsto (0,1)$, $(3,5) \mapsto (1,0) \mapsto (3,2)$, and $(2,3) \mapsto (0,1) \mapsto (4,5)$. But:

$$P^{-1}(\underline{x}) = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \underline{x} + \begin{pmatrix} -3 \\ 5 \end{pmatrix}$$

Therefore:

$$Q \circ P^{-1}(\underline{x}) = \begin{pmatrix} 3 & 4 \\ 1 & 4 \end{pmatrix} \left(\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \underline{x} + \begin{pmatrix} -3 \\ 5 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -11 & 5 \\ -17 & 7 \end{pmatrix} \underline{x} + \begin{pmatrix} 11 \\ 18 \end{pmatrix}$$

Problem 2 Let $ABCD$ be a parallelogram. Let X and Y be trisection points of the diagonal AC (where X is closer to A and Y is closer to C). Let $R = BX \cap AD$ and $S = BY \cap CD$. Let f be an affine transformation that maps B to $B' = (0,0)$, A to $A' = (3,0)$, and C to $C' = (0,3)$.

(a) Find the co-ordinates of the images D', X', Y', R' and S' of D, X, Y, R, and S under f. Give brief reasons for your answers.

Answer: Diagrams of the original and transformed configurations have been omitted, but should be drawn.

$D' = (3,3)$ (to complete a parallelogram with B', A', and C').

$X' = (2,1)$ and $Y' = (1,2)$ (as trisection points).

$R' = (3, 3/2)$ (the point on $B'X'$ with an x co-ordinate of 3).

$S' = (3/2, 3)$ (similarly).

(b) Find the slopes of $A'C'$ and $R'S'$.

Answer: Slope $A'C' = \frac{-3}{3} = -1$.

Slope $R'S' = \frac{-3/2}{3/2} = -1$.

(c) What can you say about the type of the quadrilateral $ACSR$? (Is it a square, rectangle, parallelogram, ...).

Answer: The lines $A'C'$ and $R'S'$ are parallel (same slope). Therefore, AC and RS are parallel. Hence, $ACSR$ is a trapezoid.

Problem 3 Prove the following theorem. (Hint: Use an affine transformation to an easier configuration.)

If a trapezoid is inscribed in an ellipse, then the line joining the midpoints of the two parallel sides of the trapezoid passes through the center of the ellipse.

Answer: Diagrams of the original and transformed configurations have been omitted, but should be drawn.

Suppose that a trapezoid $ABCD$ (with parallel sides AB and CD) is inscribed in an ellipse with center O. Select an affine transformation f that turns the ellipse into a circle and turns A, B, C, D, O into points A', B', C', D', O'. The chords $A'B'$ and $C'D'$ are parallel. Hence, both are perpendicular to the same diameter, d say, of the circle. This diameter d passes through O' and the midpoints X' and Y' of the two parallel chords. Apply f^{-1}: The points O, $X = f^{-1}(X')$ and $Y = f^{-1}(Y')$ are collinear. But X and Y are the midpoints of AB and CD (since midpoints are affine invariants).

Problem 4 Which of the following geometrical configurations are affine concepts? Answer with yes for any affine concepts and no for non-concepts.

(a) parallelogram **Answer:** Yes
(b) trapezoid **Answer:** Yes
(c) rectangle **Answer:** No
(d) isosceles trapezoid **Answer:** No
(e) conic **Answer:** Yes
(f) ellipse **Answer:** Yes
(g) ratio of perpendicular segments **Answer:** No
(h) ratio of parallel segments **Answer:** Yes
(i) incircle of a triangle **Answer:** No
(j) collinear triple of points, one on each of three sides of a triangle **Answer:** Yes

Problem 5 Let ABC be a triangle. Let BD be a median of triangle ABC. Let X and Y be two points on the segment BC such that $BX = XY = YC$. (So X and Y divide BC into 3 equal parts.) Let AX meet BD at Z. Use affine geometry to prove that Z is the midpoint of BD.

Answer: Let A', B', C', D', X', Y', and Z' be the images of A, B, C, D, X, Y, Z under an affine transformation that is selected so that $B' = (0, 0)$, $A' = (2, 0)$, $C' = (0, 6)$.

We have $D' = (1, 3)$, $X' = (0, 2)$, $Y' = (0, 4)$ (since affine transformations preserve ratios along each line).

Moreover, Z' is the intersection of the lines joining A' to X' and B' to D'. These are the lines $x + y = 2$ and $y = 3x$. Their intersection is the point $(1/2, 3/2)$. This is also the midpoint of $B'D'$. Hence Z' is the midpoint of $B'D'$. Therefore Z is the midpoint of BD (since midpoints are affine invariants).

Problem 6 Let $ABCD$ be a parallelogram. Let X and Y be the midpoints of the sides AB and BC. Let AY meet CX at Z. Use affine geometry to prove that Z lies on the diagonal BD and to determine the ratio DZ/ZB.

Answer: Let A', B', C', D', X', Y', and Z' be the images of A, B, C, D, X, Y, Z under an affine transformation that is selected so that $B' = (0, 0)$, $A' = (2, 0)$, $C' = (0, 2)$.

We have $D' = (2, 2)$ since affine transformations preserve parallelograms. Also $X' = (1, 0)$, $Y' = (0, 1)$ (since affine transformations preserve ratios along each line).

Moreover, Z' is the intersection of the lines joining A' to Y' and C' to X'. These are the lines $y = -x/2 + 1$ and $y = -2x + 2$. Their intersection is the point $(2/3, 2/3)$. This is a point of the line $B'D'$ (which has equation $y = x$). Also $D'Z'/Z'B' = 2$. Therefore Z lies on the segment BD with $DZ/ZB = 2$ (since ratios along a segment are affine invariants).

Problem 7 Let ABC be a triangle. Let BD be a median of triangle ABC. Let X, Y, and Z be three points on the segment BC such that $BX = XY = YZ = ZC$. (So X, Y and Z divide BC into 4 equal parts.) Let AX meet BD at P. Use affine geometry to prove that $AP/AX = 4/5$.

Answer: Let A', B', C', D', P', X', Y', and Z' be the images of A, B, C, D, P, X, Y, Z under an affine transformation that is selected so that $B' = (0, 0)$, $A' = (2, 0)$, $C' = (0, 8)$.

We have $D' = (1, 4)$, $X' = (0, 2)$, $Y' = (0, 4)$, $Z' = (0, 6)$ (since affine transformations preserve ratios along each line).

Moreover, P' is the intersection of the lines joining A' to X' and B' to D'. These are the lines $x + y = 2$ and $y = 4x$. Their intersection is the point $(2/5, 8/5)$. This divides the segment $A'X'$ in the ratio $A'P'/A'X' = 4/5$. Hence $AP/AX = 4/5$ (since ratios along a line are affine invariants).

Problem 8 Let ABC be a triangle and let D be a point on the side BC. Let X and Y be the centroids of triangles ABD and ACD. Use affine geometry to prove that the line XY is parallel to BC.

Answer: Let A', B', C', D', X', and Y' be the images of A, B, C, D, X, Y under an affine transformation that is selected so that $A' = (0, 0)$, $B' = (-6, 3)$, $D' = (0, 3)$.

We have $C' = (6a, 3)$, for some value of a, since B, D, and C are collinear. We have $X' = (-2, 2)$ since X is 2/3 of the way from A to the midpoint of BD. Similarly, $Y' = (2a, 2)$. The lines $B'C'$ and $X'Y'$ are parallel (because both are horizontal). Therefore BC is parallel to XY.

Problem 9 Let T be a trapezoid. Prove that there is an affine transformation that transforms T to an isosceles trapezoid. (An isosceles trapezoid is a trapezoid with two equal base angles.)

Answer: Write A and D for the base vertices of the trapezoid and write AB and DC for the two non-parallel opposite sides. Let X be the intersection point of AB and BC.

There is an affine transformation that transforms AXD to an equilateral triangle $A'X'D'$. Let B' and C' be the images of B and C under this transformation. Then $A'D'$ is parallel to $B'C'$ (because AD is parallel to BC). Therefore $A'B'C'D'$ is a trapezoid. However its base angles are equal because they coincide with angles of the equilateral triangle $A'X'D'$.