

Problem 1 (a) Write the two forms of the hyperbolic law of cosines.

Answer:

$$\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(C)$$

$$\cos(C) = -\cos(A)\cos(B) + \sin(A)\sin(B)\cosh(c)$$

(b) Find the hyperbolic length of the shortest side of a hyperbolic triangle with angles 60° , 45° , and 30° . (Which form of the law of cosines is helpful here?)

Answer:

Apply the second form of the law of cosines. Use $C = 30$, $A = 45$, $B = 60$. The shortest side is c . (The shortest side is opposite the smallest angle.)

We have $\cos(30) = -\cos(45)\cos(60) + \sin(45)\sin(60)\cosh(c)$. Hence: $\sqrt{3}/2 = -1/(2\sqrt{2}) + \sqrt{3}\cosh(c)/(2\sqrt{2})$ and $c = \cosh^{-1}(\sqrt{2} + 1/\sqrt{3})$.

Problem 2 Find a formula for $F(z)$ where F is a Mobius transformation with $F(1) = \infty$, $F(2) = 1$, and $F(i) = 3$. (Write your answer as a function $F(z)$ that has the form $\frac{az+b}{cz+d}$.)

Answer:

Let f be a Mobius transformation with $f(1) = \infty$, $f(2) = 0$, $f(i) = 1$. Then $F(z) = 2f(z) + 1$ is the required transformation.

Now:

$$f(z) = \frac{(z-2)(i-1)}{(z-1)(i-2)}.$$

$$\text{And therefore: } F(z) = \frac{(3i-4)z+(6-5i)}{(i-2)z+(2-i)}$$

Problem 3 (i) Define a hyperbolic circle.

Answer: A hyperbolic circle has a center and a radius. It is the locus of all (hyperbolic) points whose hyperbolic distance from the center is equal to the radius.

(ii) State the origin lemma of hyperbolic geometry.

Answer: If O and P are any hyperbolic points then there is a hyperbolic reflection that interchanges O and P .

(iii) Prove that every hyperbolic circle is represented by a Euclidean circle (when we use the standard unit disk model of the hyperbolic plane).

Answer:

Suppose that a hyperbolic circle C has center at P . Let O be the center of the unit circle. Let t be a hyperbolic reflection that maps P to O . Then $t(C)$ is a hyperbolic circle centered at O . However, the hyperbolic distance from a point to O is a function of its Euclidean distance. Therefore, $t(C)$ is a Euclidean circle centered at O . Hence C is the image under $t^{-1} = t$ of a Euclidean circle. However, the hyperbolic reflection t is represented by an inversion, and therefore the t -image of a Euclidean circle is either a circle or a line. In particular C is either a (Euclidean) circle or a line. However, C is entirely contained in the unit disk, and must therefore be a Euclidean circle.

Problem 4 Suppose that ABC is a triangle and that D , E , and F are trisection points that divide the sides BC , CA and AB in the ratio $1 : 2$. (Thus $\frac{BD}{DC} = \frac{1}{2}$, etc.).

(i) Explain why there must be an affine transformation that maps all of the points A, B, C, D, E, F to points with integer co-ordinates. Write down a possible set of integer co-ordinates for their images. (For example, you could choose to let $A' = (0, 0)$. You must specify the images of the other points under your chosen affine transformation.)

Answer: There is an affine transformation that transforms A, B , and C to $(0, 0)$, $(3, 0)$, and $(0, 3)$. The trisection points D, E , and F are then transformed to $(2, 1)$, $(0, 2)$, $(1, 0)$.

(ii) Suppose that AD meets EF at the point X . Determine the ratio $EX : XF$.

Answer:

The line $E'F'$ has equation: $y = -2x + 2$.

The line $A'D'$ has equation: $2y = x$.

These lines meet at $(4/5, 2/5)$. This is the point X' .

By comparing x -coordinates, $E'X' : X'F' = \frac{4/5}{1/5}$. Therefore, since ratios along a line are affine invariant, we have $EX : XF = \frac{4}{1}$.

Problem 5 Prove that a pair of angle bisectors in a hyperbolic triangle intersect a (hyperbolic) point whose perpendicular distances from the three sides are equal. (The perpendicular distance from a point to a line is the hyperbolic length of the perpendicular from the point to the line.)

Answer: Suppose that the vertices of the hyperbolic triangle are A , B , and C . Suppose that the bisectors of the angles at A and C meet at X . Let D , E , and F be the feet of perpendiculars from X to the sides BC , CA , and AB . Then the hyperbolic triangles AFX and AEX are congruent (AAS). Therefore $d_h(F, X) = d_h(E, X)$ (CPCTC). Similarly, triangles CDX and CEX are congruent and $d_h(E, X) = d_h(D, X)$.

Note: this proof is identical to the proof of the analogous result in Euclidean geometry.

Problem 6 Suppose that P is a point outside the circle Σ . Let O be the center of Σ . Let the tangents from P meet Σ at X and Y . Let XY meet OP at Z .

Prove that P and Z are inverse points with respect to Σ .

Answer: Triangles OXP and OZX are similar (the angles at O are common, and the two triangles also have right angles, at X and Z). Therefore: $\frac{OX}{OP} = \frac{OZ}{OX}$. We deduce that $OP \cdot OZ = OX^2$. In other words, P and Z are inverses across Σ .