Solutions to exam 1: 3-19-2008

Problem 1 Find an affine transformation that turns the points $(-1,-1)$, $(1,2)$, $(2,3)$ into the points $(0,1)$, $(1,3)$, $(2,2)$, respectively. (Give a formula $f(\underline{x}) = M\underline{x} + \underline{b}$ for the transformation.)

Answer:

The transformation that maps $(0,0)$, $(1,0)$, $(0,1)$ to $(-1,-1)$, $(1,2)$, $(2,3)$ is:

$$P(\underline{x}) = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \underline{x} + \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Similarly, the transformation that maps $(0,0)$, $(1,0)$, $(0,1)$ to $(0,1)$, $(1,3)$, $(2,2)$ is:

$$Q(\underline{x}) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \underline{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence, $Q \circ P^{-1}$ maps $(-1,-1) \mapsto (0,0) \mapsto (0,1)$, $(1,2) \mapsto (1,0) \mapsto (1,3)$, and $(2,3) \mapsto (0,1) \mapsto (2,2)$. But:

$$P^{-1}(\underline{x}) = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix} \underline{x} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Therefore:

$$Q \circ P^{-1}(\underline{x}) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \left(\begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix} \underline{x} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -5 & 4 \end{pmatrix} \underline{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Problem 2 An ellipse is inscribed in a quadrilateral so as to touch all four sides at their midpoints. Prove that the quadrilateral is a parallelogram.

Answer: Suppose that the ellipse touches the sides AB, BC, CD, and DA of the quadrilateral at W, X, Y, and Z. Choose an affine transformation that turns A, B, C, D, W, X, Y, Z to points $A', B', C', D', W', X', Y', Z'$ and turns the ellipse into a circle. (Omitted diagrams showing the original and transformed configurations should be drawn.)

We have $A'B' = A'W' + W'B' = A'Z' + B'X'$ (by the equal tangents theorem). Similarly, $D'C' = D'Z' + C'X'$. However, midpoints are affine invariant, so that $A'Z' = D'Z'$ and $B'X' = C'X'$. Therefore $A'B' = D'C'$. Similarly, $A'D' = C'B'$.

This means that $A'B'C'D'$ must be a parallelogram (two pairs of opposite sides are equal). But parallelograms are affine invariant, and therefore $ABCD$ is also a parallelogram.

Problem 3 Find a formula for $F(z)$ where F is a Mobius transformation with $F(0) = -2i$, $F(2) = 2$, and $F(1-i) = 1-i$. (Write your answer as a function $F(z)$ that has the form $\frac{az+b}{cz+d}$.)

Answer:

Let f be a Mobius transformation with $f(-2i) = 0$, $f(1-i) = 1$, $f(2) = \infty$. Let g be a Mobius transformation with $g(0) = 0$, $g(1-i) = 1$, $g(2) = \infty$. Then $F(z) = f^{-1}(g(z))$ is the required transformation.

Now:

$f(z) = \frac{(z+2i)(-1-i)}{(z-2)(1+i)} = \frac{-z-2i}{z-2}$. Hence f has matrix:

$$\begin{pmatrix} -1 & -2i \\ 1 & -2 \end{pmatrix}$$

And: $g(z) = \frac{(z)(-1-i)}{(z-2)(1-i)} = \frac{-iz}{z-2}$. Hence g has matrix:

$$\begin{pmatrix} -i & 0 \\ 1 & -2 \end{pmatrix}$$

Hence F has matrix:

$$\begin{pmatrix} -2 & 2i \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -i & 0 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 4i & -4i \\ i-1 & 2 \end{pmatrix}$$

Thus $F(z) = \frac{4iz-4i}{(i-1)z+2}$.

Problem 4 (a) Do the four points 2, -2, $2+3i$, and $1+3i$ lie on a circle. (You must explain your answer to get any credit!)

Answer: Consider the Mobius transformation $F(z) = \frac{(z-2)((2+3i)+2)}{(z+2)((2+3i)-2)} = \frac{(z-2)(4+3i)}{(z+2)3i}$.

$F(z)$ transforms the unique generalized circle through the three points 2, -2, $2+3i$ to the real axis.

But: $F(1+3i) = \frac{9i-13}{9i-9}$ which is not real. Therefore $1+3i$ does not lie on the only (generalized) circle that can pass through the other 3 points.

(b) Describe the image curve that is obtained when the y-axis (the imaginary axis) is transformed by using the Mobius transformation:

$$M(z) = \frac{-iz+1}{z+1}$$

(You must completely describe the image curve. It is not enough just to say that it is a line or to say that it is a circle. You should give details of the slope, center or other features that completely describe the curve.)

Answer:

The image is a generalized circle.

However, $M(0) = 1$, $M(\infty) = -i$, $M(-i) = 0$. These points lie on the image curve, which must therefore be the circle with center $(1-i)/2$ and radius $1/\sqrt{2}$.

Problem 5 Suppose that P and Q are points. Prove that the locus of all points R with $QR/PR = 2$ is a generalized circle.

Answer: Consider an inversion with center P. Write Q' and R' for the inverses of the fixed point Q and the variable point R.

Then, triangles PQR and $PR'Q'$ are similar. Hence, $2 = \frac{QR}{PR} = \frac{R'Q'}{PQ'}$. Thus, $R'Q' = 2PQ'$, a constant. We deduce that as R moves, its inverse moves on a circle centered at Q'. It follows that R moves on a cruve that is the inverse of a circle, which must be a generalized circle.